RECOGNISING ACHIEVEMENT

GCE

Mathematics (MEI)

Advanced GCE

Unit **4756:** Further Methods for Advanced Mathematics

Mark Scheme for January 2013

Oxford Cambridge and RSA Examinations

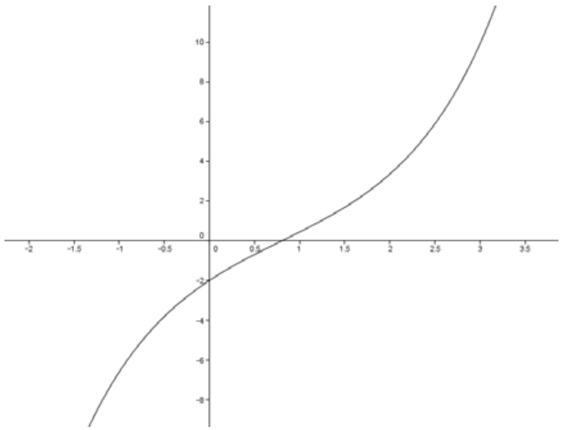
Question			Answer	Marks	Guidance	
1	(a)	(i)	$a\tan y = x \Rightarrow a\sec^2 y\dfrac{dy}{dx} = 1$	M1	Differentiating with respect to x or y	$\dfrac{dx}{dy} = a\sec^2 y$
			$\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{a\sec^2 y}$	A1	For $\dfrac{dy}{dx}$	Or $a\dfrac{dy}{dx} = \dfrac{1}{\sec^2 y}$
			$\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{a\left(1+\dfrac{x^2}{a^2}\right)}$			
			$\Rightarrow \dfrac{dy}{dx} = \dfrac{a}{a^2+x^2}$	A1(ag)	Completion www with sufficient detail	
				[3]		
1	(a)	(ii)	$x^2 - 4x + 8 = (x-2)^2 + 4$	B1		
			$\int_0^4 \dfrac{1}{x^2-4x+8}dx = \dfrac{1}{2}\left[\arctan\dfrac{x-2}{2}\right]_0^4$	M1 A1	Integral of form a arctan bu or any appropriate substitution Correct integral with consistent limits	$\dfrac{1}{2}\left[\arctan\dfrac{u}{2}\right]_{-2}^{2}$
			$= \dfrac{1}{2}(\arctan(1) - \arctan(-1))$			
			$= \dfrac{\pi}{4}$	A1	Evaluated in terms of π	
				[4]		
1	(a)	(iii)	$\int 1\times\arctan x\,dx$	M1	Using parts with u = arctan x and v' = 1	Allow one other error
			$= x\arctan x - \int\dfrac{x}{1+x^2}dx$	A1		
				M1	$\int\dfrac{x}{1+x^2}dx = a\ln(1+x^2)$	
			$= x\arctan x - \dfrac{1}{2}\ln(1+x^2) + c$	A1	$a = \dfrac{1}{2}$. Condone omitted c	
				[4]		

Question			Answer	Marks	Guidance	
1	(b)	(i)	$r = 2\cos\theta \Rightarrow r^2 = 2r\cos\theta$	M1	Using $r^2 = x^2 + y^2$ and $x = r\cos\theta$	
			$\Rightarrow x^2 + y^2 = 2x$	A1	A correct cartesian equation in any form	
			$\Rightarrow (x-1)^2 + y^2 = 1$	A1(ag)	Explaining that the curve is a circle	e.g. writing as $(x-\alpha)^2 + (y-\beta)^2 = r^2$
			OR $x = r\cos\theta \Rightarrow x = 2\cos^2\theta$ $y = r\sin\theta \Rightarrow y = 2\cos\theta\sin\theta = \sin 2\theta$ M1 $\cos 2\theta = 2\cos^2\theta - 1 \Rightarrow x = \cos 2\theta + 1$ A1 $\Rightarrow (x-1)^2 + y^2 = 1$ A1(ag)		Using $x = r\cos\theta$, $y = r\sin\theta$ and linking x in terms of $\cos 2\theta$ Explaining that the curve is a circle	e.g. writing as $(x-\alpha)^2 + (y-\beta)^2 = r^2$
			Centre (1, 0) Radius 1	B1 B1 **[5]**	Independent Independent	
1	(b)	(ii)	$x^2 + (y-2)^2 = 4 \Rightarrow x^2 + y^2 = 4y$ $\Rightarrow r^2 = 4r\sin\theta$ $\Rightarrow r = 4\sin\theta$	 M1 A1 **[2]**	 Using $r^2 = x^2 + y^2$ and $y = r\sin\theta$	 For answer alone www: B1 for $r = k\sin\theta$, B1 for $k = 4$
2	(a)	(i)	$1 + e^{j2\theta} = 1 + \cos 2\theta + j\sin 2\theta$ $= 1 + (2\cos^2\theta - 1) + 2j\sin\theta\cos\theta$ $= 2\cos^2\theta + 2j\sin\theta\cos\theta$ $= 2\cos\theta(\cos\theta + j\sin\theta)$	 M1 A1(ag)	 Using $e^{2j\theta} = \cos 2\theta + j\sin 2\theta$ and double angle formulae Completion www	 Allow one error
			OR $1 + e^{j2\theta} = e^{j\theta}(e^{-j\theta} + e^{j\theta})$ M1 $= (\cos\theta + j\sin\theta) \times 2\cos\theta$ A1(ag)		"Factorising" and complete replacement by trigonometric functions Completion www	
			OR $1 + e^{j2\theta} = 1 + (\cos\theta + j\sin\theta)^2$ $= 1 + \cos^2\theta - \sin^2\theta + 2j\sin\theta\cos\theta$ $= 2\cos^2\theta + 2j\sin\theta\cos\theta$ M1 $= 2\cos\theta(\cos\theta + j\sin\theta)$ A1(ag)		Using $e^{j\theta} = \cos\theta + j\sin\theta$ and $1 - \sin^2\theta = \cos^2\theta$ Completion www	
				[2]		

Question			Answer	Marks	Guidance	
2	(a)	(ii)	$C+\mathrm{j}S=1+\binom{n}{1}e^{\mathrm{j}2\theta}+\binom{n}{2}e^{\mathrm{j}4\theta}+\ldots+e^{\mathrm{j}2n\theta}$	M1	Forming $C+\mathrm{j}S$	
			$=\left(1+e^{\mathrm{j}2\theta}\right)^n$	M1 A1	Recognising as binomial expansion	
			$=2^n\cos^n\theta\left(\cos\theta+\mathrm{j}\sin\theta\right)^n$			
			$=2^n\cos^n\theta\left(\cos n\theta+\mathrm{j}\sin n\theta\right)$	M1 A1	Applying (i) and De Moivre o.e.	Dependent on M1M1 above
			$\Rightarrow C=2^n\cos^n\theta\cos n\theta$	A1(ag)	Completion www	Need to see $e^{\mathrm{j}n\theta}=\cos n\theta+\mathrm{j}\sin n\theta$ o.e.
			and $S=2^n\cos^n\theta\sin n\theta$	A1 **[7]**		
2	(b)	(i)	$e^{\mathrm{j}\frac{2\pi}{3}}=\cos\frac{2\pi}{3}+\mathrm{j}\sin\frac{2\pi}{3}=-\frac{1}{2}+\mathrm{j}\frac{\sqrt{3}}{2}$	B1 **[1]**	Must evaluate trigonometric functions	
2	(b)	(ii)	Other two vertices are $(2+4\mathrm{j})e^{\mathrm{j}\frac{2\pi}{3}}$	M1	Award for idea of rotation by $\frac{2\pi}{3}$	e.g. use of $\arctan 2+\frac{2\pi}{3}$ (3.202 rad) (must be 2)
			$=(2+4\mathrm{j})\left(-\frac{1}{2}+\mathrm{j}\frac{\sqrt{3}}{2}\right)$			
			$=\left(-1-2\sqrt{3}\right)+\mathrm{j}\left(-2+\sqrt{3}\right)$	A1A1	May be given as co-ordinates	
			and $(2+4\mathrm{j})e^{\mathrm{j}\frac{4\pi}{3}}=(2+4\mathrm{j})e^{-\mathrm{j}\frac{2\pi}{3}}$	M1	Award for idea of rotation by $-\frac{2\pi}{3}$	e.g. use of $\arctan 2+\frac{4\pi}{3}$ (5.296 rad) (must be 2)
			$=(2+4\mathrm{j})\left(-\frac{1}{2}-\mathrm{j}\frac{\sqrt{3}}{2}\right)$			
			$=\left(-1+2\sqrt{3}\right)+\mathrm{j}\left(-2-\sqrt{3}\right)$	A1A1 **[6]**	May be given as co-ordinates	If A0A0A0A0 award SC1 for awrt −4.46 − 0.27j and 2.46 − 3.73j

Question			Answer	Marks	Guidance	
2	(b)	(iii)	Length of (2 + 4j) = $\sqrt{20}$ So length of side = $2\sqrt{20}\cos\frac{\pi}{6} = 2\sqrt{20}\times\frac{\sqrt{3}}{2}$ $= 2\sqrt{15}$	M1 A1(ag) **[2]**	Complete method Completion www	Alternative: finding distance between (2, 4) and $\left(-1-2\sqrt{3}, -2+\sqrt{3}\right)$ o.e.
3	(i)		$\mathbf{M}-\lambda\mathbf{I} = \begin{pmatrix} 1-\lambda & 3 & 0 \\ 3 & -2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{pmatrix}$ $\det(\mathbf{M}-\lambda\mathbf{I})$ $=(1-\lambda)\left[(-2-\lambda)(1-\lambda)-1\right]-3\left[3(1-\lambda)\right]$ $=(1-\lambda)\left(\lambda^2+\lambda-3\right)-9(1-\lambda)$ $\Rightarrow \lambda^3-13\lambda+12=0$	M1 A1 A1(ag) **[3]**	Forming $\det(\mathbf{M}-\lambda\mathbf{I})$ Any correct form Condone omission of 0	Sarrus: $(1-\lambda)^2(-2-\lambda)-10(1-\lambda)$ or e.g. $\lambda-1+(1-\lambda)\left(\lambda^2+\lambda-11\right)$
3	(ii)		$(\lambda-1)\left(\lambda^2+\lambda-12\right)=0$ $\Rightarrow(\lambda-1)(\lambda-3)(\lambda+4)=0$ $\Rightarrow$ eigenvalues are 1, 3, −4 $\lambda=1: \begin{pmatrix} 0 & 3 & 0 \\ 3 & -3 & -1 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\Rightarrow y=0, 3x-z=0$ $\Rightarrow$ eigenvector is $\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$ $\lambda=3: \begin{pmatrix} -2 & 3 & 0 \\ 3 & -5 & -1 \\ 0 & -1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\Rightarrow -2x+3y=0, -y-2z=0$	M1 A1 A1 M2 M1 A1 A1 A1	Factorising as far as quadratic <u>For any one of λ = 1, 3, −4</u> Obtaining two independent equations Obtaining a non-zero eigenvector o.e. o.e.	Allow one error From which an eigenvector could be found Allow e.g. $3y=0, 3x-3y-z=0$

Question			Answer	Marks	Guidance	
			$\Rightarrow y = -2z,\ x = -3z$			
			$\Rightarrow$ eigenvector is $\begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix}$	A1		
			$\lambda = -4:\ \begin{pmatrix} 5 & 3 & 0 \\ 3 & 2 & -1 \\ 0 & -1 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$			
			$\Rightarrow 5x + 3y = 0,\ -y + 5z = 0$	A1	o.e.	
			$\Rightarrow y = 5z,\ x = -3z$			
			$\Rightarrow$ eigenvector is $\begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix}$	A1		
				[12]		
3	**(iii)**		E.g. $\mathbf{P} = \begin{pmatrix} 1 & -3 & -3 \\ 0 & -2 & 5 \\ 3 & 1 & 1 \end{pmatrix}$	B1	Use of eigenvectors (ft) as columns	
			$\mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3^n & 0 \\ 0 & 0 & (-4)^n \end{pmatrix}$	M1	Use of 1, 3, −4 (ft) in correct order	n not required for M1
				A1	Power n	-4^n A0
				[3]		

Question			Answer	Marks	Guidance	
4	**(i)**		$y = 3\sinh x - 2\cosh x$			$\frac{1}{2}e^{x} - \frac{5}{2}e^{-x}$
			$\Rightarrow \frac{dy}{dx} = 3\cosh x - 2\sinh x$	B1		$\frac{1}{2}e^{x} + \frac{5}{2}e^{-x}$
			At TPs, $\frac{dy}{dx} = 0 \Rightarrow \tanh x = \frac{3}{2}$	M1	Considering $\frac{dy}{dx} = 0$	$e^{2x} = -5$; $e^{x} > 0$ and $e^{-x} > 0$
			which has no (real) solutions	A1(ag)	Showing no real roots www	
			$y = 0 \Rightarrow \tanh x = \frac{2}{3}$	M1	Solving $y = 0$ as far as e^{2x} or $\tanh x$ etc.	$e^{2x} = 5$; $\cosh x = \frac{3}{\sqrt{5}}$; $\sinh x = \frac{2}{\sqrt{5}}$
			$\Rightarrow x = \frac{1}{2}\ln\frac{1+\frac{2}{3}}{1-\frac{2}{3}}$	M1	Solving as far as x	<u>Attempt to verify</u> Award M1 for substituting $x = \frac{1}{2}\ln 5$
			$\Rightarrow x = \frac{1}{2}\ln 5$	A1(ag)	Completion www	and M1 for clearly attempting to evaluate exactly
			$\frac{d^2y}{dx^2} = 3\sinh x - 2\cosh x = y$			
			so $y = 0 \Rightarrow \frac{d^2y}{dx^2} = 0$	B1(ag) **[7]**		$3\sinh\left(\frac{1}{2}\ln 5\right) - 2\cosh\left(\frac{1}{2}\ln 5\right) = 0$ must be explained, e.g. connected with $y = 0$
4	**(ii)**			B2 **[2]**	For a curve with the following features: • increasing • intersecting the positive x-axis • $(0, -2)$ indicated • gradient increasing with large $\lvert x \rvert$ • one point of inflection	Award B1 for a curve lacking one of these features

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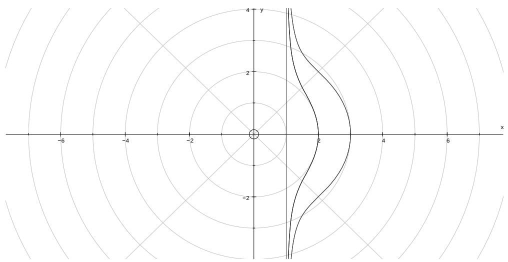
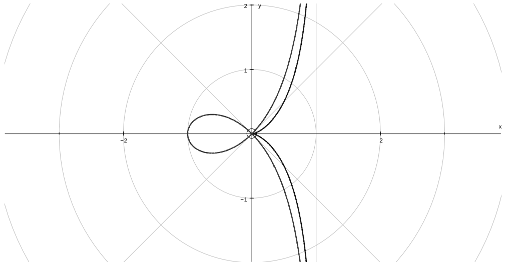
4	(iii)		$(3\sinh x - 2\cosh x)^2$ $= 9\sinh^2 x - 12\sinh x\cosh x + 4\cosh^2 x$	B1		
			$= \frac{9}{2}(\cosh 2x - 1) - 6\sinh 2x + 2(\cosh 2x + 1)$	M1	Using double "angle" formulae or complete alternative	Condone sign errors but need $\frac{1}{2}$ s
			$= \frac{13}{2}\cosh 2x - 6\sinh 2x - \frac{5}{2}$	A1	Accept unsimplified	$\frac{1}{4}e^{2x} + \frac{25}{4}e^{-2x} - \frac{5}{2}$
			$V = \pi\int_0^{\frac{1}{2}\ln 5} y^2 \mathrm{d}x$	M1	Attempting to integrate their y^2 (ignore limits)	
			$= \pi\left[\frac{13}{4}\sinh 2x - 3\cosh 2x - \frac{5}{2}x\right]_0^{\frac{1}{2}\ln 5}$	A2	Correct results and limits c.a.o. Ignore omitted π	Give A1 for one error, or for all three terms correct and incorrect limits
			$= \pi\left[\frac{13}{4}\times\frac{12}{5} - 3\times\frac{13}{5} - \frac{5}{4}\ln 5 + 3\right]$	M1 M1	Substituting both of their limits Obtaining exact values of $\sinh(\ln 5)$ and $\cosh(\ln 5)$	$\sinh(\ln 5) = \frac{12}{5}$, $\cosh(\ln 5) = \frac{13}{5}$
			OR $= \pi\left[\frac{1}{8}e^{2x} - \frac{25}{8}e^{-2x} - \frac{5}{2}x\right]_0^{\frac{1}{2}\ln 5}$ A2		Correct results and limits	Give A1 for one error, or for all three terms correct and incorrect limits
			$= \pi\left[\frac{5}{8} - \frac{5}{8} - \frac{5}{4}\ln 5 + 3\right]$ M1 M1		Substituting both of their limits Obtaining exact values of e^{2x} and e^{-2x}	$e^{2x} = 5$, $e^{-2x} = \frac{1}{5}$
			$= \pi\left[3 - \frac{5}{4}\ln 5\right]$	A1(ag) **[9]**	Completion www	
5	(i)			B2 B1 **[3]**	Three curves of correct shape Correctly identified	Give B1 for two correct curves $a = 0$, $a = 1$, $a = 2$ from left to right

Question			Answer	Marks	Guidance	
5	(ii)			B1 B1 **[2]**	Curve for $a = -1$ Curve for $a = -2$	Curve with cusp Curve with loop
5	(iii)		Asymptote	B1 **[1]**		
5	(iv)		$a = -1$: cusp $a = -2$: loop	B1 B1 **[2]**		
5	(v)		$r = \sec\theta + a\cos\theta \Rightarrow r\cos\theta = 1 + a\cos^2\theta$ $\Rightarrow x = 1 + a\left(\frac{x^2}{r^2}\right)$ $\Rightarrow x - 1 = a\left(\frac{x^2}{x^2 + y^2}\right)$ $\Rightarrow x^2 + y^2 = a\left(\frac{x^2}{x-1}\right) \Rightarrow y^2 = a\left(\frac{x^2}{x-1}\right) - x^2$ Hence asymptote at $x = 1$	 M1 M1 M1 A1(ag) B1 **[5]**	 Using $x = r\cos\theta$ Using $r^2 = x^2 + y^2$ Making y^2 subject	
5	(vi)		Curve exists for $y^2 \geq 0$ $\Rightarrow a\left(\frac{1}{x-1}\right) - 1 \geq 0$ If $a > 0$ then $x - 1 > 0$ and so $a \geq x - 1$ i.e. $1 < x \leq 1 + a$ If $a < 0$ then $x - 1 < 0$ and so $a \leq x - 1$ i.e. $1 + a \leq x < 1$	 M1 M1 A1(ag) M1 A1 **[5]**	 Considering $y^2 \geq 0$	